

# THE RATE OF GROWTH OF MOMENTS OF CERTAIN COTANGENT SUMS

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**ABSTRACT.** We consider cotangent sums associated to the zeros of the Estermann zeta function considered by the authors in their previous paper [5]. We settle a question on the rate of growth of the moments of these cotangent sums left open in [5], and obtain a simpler proof of the equidistribution of these sums.

**Key words:** Cotangent sums; equidistribution; Estermann zeta function; moments; continued fractions; measure.

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## 1. INTRODUCTION

The authors in joint work and the second author in his thesis, investigated the distribution of cotangent sums

$$c_0\left(\frac{r}{b}\right) = - \sum_{m=1}^{b-1} \frac{m}{b} \cot\left(\frac{\pi mr}{b}\right),$$

as  $r$  ranges over the set

$$\{r : (r, b) = 1, A_0 b \leq r \leq A_1 b\},$$

where  $A_0, A_1$  are fixed with  $1/2 < A_0 < A_1 < 1$  and  $b$  tends to infinity. Especially, they considered the moments

$$H_k = \lim_{b \rightarrow +\infty} \phi(b)^{-1} b^{-2k} (A_1 - A_0)^{-1} \sum_{\substack{A_0 b \leq r \leq A_1 b \\ (r, b) = 1}} c_0\left(\frac{r}{b}\right)^{2k}, \quad k \in \mathbb{N},$$

where  $\phi(\cdot)$  denotes the Euler phi-function.

They could show that all the moments  $H_k$  exist and that

$$\lim_{k \rightarrow +\infty} H_k^{1/k} = +\infty$$

Thus the series  $\sum_{k \geq 0} H_k x^{2k}$  converges only for  $x = 0$ .

It was left open, whether the series

$$(*) \quad \sum_{k \geq 0} \frac{H_k}{(2k)!} x^k$$

converges for values of  $x$  different from 0. This fact would considerably simplify the proof for the distribution of the cotangent sums  $c_0(r/b)$  (uniqueness of measures determined by their moments, see [1], Section 30, The Method of Moments,

Theorem 30.1).

Crucial for the investigation was the result:

$$H_k = \int_0^1 \left( \frac{g(x)}{2\pi} \right)^{2k} dx,$$

where

$$g(x) = \sum_{l \geq 1} \frac{1 - 2\{lx\}}{l}.$$

The function  $g$  has been also investigated in the paper [2] of R. de la Bretèche and G. Tenenbaum. Their ideas will be crucial in our paper. We shall show the following theorems.

**Theorem 1.1.** *There exists a constant  $C_0 > 0$ , such that*

$$\int_0^1 |g(x)|^L dx \leq C_0^L L^L,$$

for all  $L \in \mathbb{N}$ .

**Theorem 1.2.** *The series*

$$\sum_{k \geq 0} \frac{H_k}{(2k)!} x^k$$

diverges for  $|x| > \pi^2$ , where  $x \in \mathbb{C}$ .

From Theorem 1.1, an *affirmative answer* regarding the question of the positive radius of convergence of (\*) follows. From Theorem 1.2 it follows that the radius of convergence of the series (\*) is finite.

**Conjecture 1.3.** *The radius of convergence of the series (\*) is  $\pi^2$ .*

## 2. CONTINUED FRACTIONS

**Definition 2.1.** *Let  $\alpha \in [0, 1] \setminus \mathbb{Q}$ . Assume that*

$$\alpha = [0; a_1, a_2, \dots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

is its continued fraction expansion with integers  $a_i \geq 1$  for  $i = 1, 2, \dots$

We denote the partial quotients by  $p_r/q_r$ , i.e.

$$[0; a_1, a_2, \dots, a_r] = \frac{p_r}{q_r}, \text{ with } (p_r, q_r) = 1.$$

We set  $p_{-1} = 1$ ,  $q_{-1} = 0$ ,  $p_0 = 0$ ,  $q_0 = 1$ .

**Definition 2.2.** *The map  $T : (0, 1) \rightarrow (0, 1)$ ,  $\alpha \mapsto \frac{1}{\alpha} - \lfloor \frac{1}{\alpha} \rfloor$  is called the continued fraction map (or Gauss map).*

**Lemma 2.3.** *The partial quotients  $p_r$ ,  $q_r$  satisfy the recursion:*

$$(1) \quad p_{r+1} = a_{r+1}p_r + p_{r-1} \quad \text{and} \quad q_{r+1} = a_{r+1}q_r + q_{r-1}.$$

*Proof.* (cf. [4], p. 7).

**Lemma 2.4.** For  $\alpha = [0; a_1, a_2, \dots, a_r, a_{r+1}, \dots]$ , we have

$$(2) \quad T^r \alpha = [0; a_{r+1}, a_{r+2}, \dots]$$

The map  $T$  preserves the measure

$$(3) \quad \omega(\mathcal{E}) = \frac{1}{\log 2} \int_{\mathcal{E}} \frac{dx}{1+x},$$

i.e.  $\omega(T(\mathcal{E})) = \omega(\mathcal{E})$ , for all measurable sets  $\mathcal{E} \subset (0, 1)$ .

*Proof.* The result (2) is well known and can be easily confirmed by direct computation. For (3) cf. [3], p. 119.  $\square$

**Lemma 2.5.** There is a constant  $A_0 > 1$ , such that

$$q_r \geq A_0^r,$$

for all  $r \in \mathbb{N}$ .

*Proof.* This is well known and easily follows from (1) of Lemma 2.3.  $\square$

**Definition 2.6.** Let  $\alpha \in (0, 1) \setminus \mathbb{Q}$ ,  $r \in \mathbb{N}$ . Then, we set

$$c(\alpha, r) = \sum_{j=0}^r \frac{\log q_{j+1}}{q_j},$$

$$c(\alpha, +\infty) = \sum_{j=0}^{+\infty} \frac{\log q_{j+1}}{q_j} \in \mathbb{R} \cup \{+\infty\}$$

We define the constant  $c_0 > 0$ , by

$$c_0 \sum_{r \geq 0} A_0^{-r/2} = \frac{1}{4}$$

and define the sequence  $(w^{(r)})$  by

$$w^{(r)} = \frac{1}{2} + c_0 \sum_{j=0}^r A_0^{-j/2}.$$

For  $z \in (0, +\infty)$ , we define

$$\mathcal{E}(z, 0) := \{\alpha \in (0, 1) \setminus \mathbb{Q} : c(\alpha, 1) \geq w^{(0)} z\}, \quad (w^{(0)} = 1/2)$$

$$\mathcal{E}(z, r) := \{\alpha \in (0, 1) \setminus \mathbb{Q} : c(\alpha, r-1) < w^{(r-1)} z, \quad c(\alpha, r) \geq w^{(r)} z\}$$

$$\mathcal{E}(z, +\infty) := \{\alpha \in (0, +\infty) \setminus \mathbb{Q} : c(\alpha, +\infty) \geq z\}.$$

**Lemma 2.7.** For  $z \in (0, +\infty)$ , it holds

$$\text{meas}(\mathcal{E}(z, +\infty)) \leq \sum_{r \geq 0} \text{meas}(\mathcal{E}(z, r)),$$

where  $\text{meas}$  stands for the Lebesgue measure.

*Proof.* Assume that  $\alpha \notin \mathcal{E}(z, r)$ , for every  $r \in \mathbb{N} \cup \{0\}$ . Then it follows by induction on  $r$ , that

$$c(\alpha, r) \leq w^{(r)} z$$

and thus

$$c(\alpha, +\infty) = \lim_{r \rightarrow +\infty} c(\alpha, r) \leq \frac{3}{4} z.$$

Therefore, if  $\alpha \in \mathcal{E}(z, +\infty)$  we have  $\alpha \in \mathcal{E}(z, r)$  for at least one value of  $r \in \mathbb{N} \cup \{0\}$ . Thus

$$\mathcal{E}(z, +\infty) \subset \bigcup_{r=0}^{+\infty} \mathcal{E}(z, r),$$

which proves Lemma 2.7 □

**Lemma 2.8.** *There are absolute constants  $z_0 > 0$  and  $c_0 > 0$ , such that*

$$\text{meas}(\mathcal{E}(z, r)) \leq \exp\left(-\frac{1}{2}c_0 A_0^{r/2} z\right),$$

for all  $z \geq z_0$ .

*Proof.* Assume that  $\alpha \in \mathcal{E}(z, r)$ . We have

$$c(\alpha, r) = c(\alpha, r-1) + \frac{\log q_{r+1}}{q_r}.$$

The inequalities

$$c(\alpha, r-1) < w^{(r-1)} z \quad \text{and} \quad c(\alpha, r) \geq w^{(r)} z,$$

imply that

$$(4) \quad \frac{\log q_{r+1}}{q_r} \geq (w^{(r)} - w^{(r-1)}) z = c_0 A_0^{-r/2} z$$

and

$$(5) \quad q_{r+1} \geq \exp\left(c_0 A_0^{-r/2} q_r z\right) \geq \exp\left(c_0 q_r^{1/2} z\right).$$

From

$$q_{r+1} = a_{r+1} q_r + q_{r-1} \leq (a_{r+1} + 1) q_r$$

we obtain

$$(6) \quad \begin{aligned} a_{r+1} &\geq q_{r+1} q_r^{-1} - 1 \geq \exp\left(c_0 q_r^{1/2} z\right) q_r^{-1} - 1 \\ &\geq \exp\left(\frac{3}{4} c_0 q_r^{1/2} z\right) \geq \exp\left(\frac{3}{4} c_0 A_0^{r/2} z\right), \end{aligned}$$

if  $z_0$  is sufficiently large.

We have for all  $w > 0$ :

$$T^r\{\alpha = [0; a_1, \dots, a_{r+1}, \dots], a_{r+1} \geq w\} = \{\alpha = [0; a_{r+1}, \dots], a_{r+1} \geq w\},$$

by Lemma 2.4.

Since  $T$  preserves the measure  $\omega$ , we have:

$$\omega\{\alpha = [0; a_1, \dots, a_{r+1}, \dots], a_{r+1} \geq w\} = \omega\{\alpha = [0; a_{r+1}, \dots], a_{r+1} \geq w\}.$$

Therefore

$$[0; a_{r+1}, \dots] \leq w^{-1}$$

and thus

$$(7) \quad \begin{aligned} \omega\{\alpha = [0; a_1, \dots, a_{r+1}, \dots], a_{r+1} \geq w\} \\ \leq \frac{1}{\log 2} \int_0^{w^{-1}} \frac{dx}{1+x} \leq 2w^{-1}. \end{aligned}$$

Applying (6) and (7) we obtain

$$(8) \quad \text{meas}(\mathcal{E}(z, r)) \leq 2w^{-1}.$$

We set in (8):

$$w = \exp\left(\frac{3}{4}c_0A_0^{r/2}z\right).$$

Then

$$\text{meas}(\mathcal{E}(z, r)) \leq \exp\left(-\frac{1}{2}c_0A_0^{r/2}z\right).$$

□

**Lemma 2.9.** *There is a constant  $c_1 > 0$ , such that*

$$\text{meas}(\mathcal{E}(z, +\infty)) \leq \exp(-c_1z), \text{ if } z \geq z_0.$$

*Proof.* This follows from Lemmas 2.7 and 2.8. □

### 3. RESULTS OF R. DE LA BRETÈCHE AND G. TENENBAUM

R. de la Bretèche and G. Tenenbaum [2] prove the following result (Théorème 4.4):

**Theorem 3.1.** *The function*

$$g(\alpha) = \sum_{l \geq 1} \frac{1 - 2\{l\alpha\}}{l}$$

*converges for  $\alpha \in \mathbb{Q}$  if and only if*

$$\sum_{r \geq 1} (-1)^r \frac{\log q_{r+1}}{q_r}$$

*converges. In this case*

$$(**) \quad g(\alpha) = - \sum_{m \geq 1} \frac{\tau(m)}{\pi m} \sin(2\pi m\alpha),$$

*where  $\tau$  stands for the divisor function.*

The following definitions are adopted from [2], p. 8.

**Definition 3.2.** *For a multiplicative function  $g$  and  $x, y$  with  $1 \leq y \leq x$  and  $\theta \in \mathbb{R}$  we denote by*

$$Z_g(x, y; \theta) := \sum_{n \in S(x, y)} g(n) \sin(2\pi\theta n),$$

*where*

$$S(x, y) = \{n \leq x : P(n) \leq y\},$$

*$P(n)$  being the largest prime factor of  $n$ .*

*We set*

$$\mu(\theta; Q) := \min_{1 \leq m \leq Q} \|m\theta\| \leq \frac{1}{Q}$$

*and*

$$q(\theta; Q) := \min\{q : 1 \leq q \leq Q, \text{ with } \|q\theta\| = \mu(\theta; Q)\},$$

*where  $\|\cdot\|$  denotes the distance to the nearest integer.*

We have:

**Lemma 3.3.** *Let  $A > 0$ . For  $x \geq 2$ ,*

$$Q_x := \frac{x}{(\log x)^{4A+24}},$$

$$q := q(\theta; Q_x), \quad a \in \mathbb{Z}, \quad (a, q) = 1,$$

$$|q\theta - a| \leq \frac{1}{Q_x}, \quad \theta_q := \theta - \frac{a}{q}, \quad \theta \in \mathbb{R},$$

*one has uniformly*

$$Z_\tau(x, x; \theta) = x(\log x) \left\{ \frac{\sin^2(\pi\theta_q x)}{\pi q \theta_q x} + O\left(\frac{(\log q) \log(1 + (\theta_q x)^2)}{q |\theta_q| x \log x}\right) + \frac{1}{(\log x)^A} \right\}$$

*Proof.* This is Lemma 11.2 of [2], pp. 64-65.  $\square$

**Definition 3.4.** *For  $\theta \in \mathbb{R} \setminus \mathbb{Q}$  let  $(q_m)_{m \geq 1} = (q_m(\theta))_{m \geq 1}$  denote the sequence of the denominators of the partial fractions of  $\theta$ . Let  $a_m/q_m$  denote the  $m$ -th partial fraction of  $\theta$ .*

*We set*

$$\varepsilon_m := \theta - \frac{a_m}{q_m}.$$

*The set of all real numbers for which  $q(\theta; Q_x) = q_m$  is an interval defined by the conditions  $q_m \leq Q_x < q_{m+1}$ . We denote it by  $[\xi_m, \xi_{m+1}]$ .*

Then, we have:

**Lemma 3.5.** *For a positive real constant  $B$ , we have:*

$$\xi_m \asymp q_m (\log q_m)^B,$$

$$|\varepsilon_m| \xi_m \asymp \frac{(\log q_m)^B}{q_{m+1}},$$

$$|\varepsilon_m| \xi_{m+1} \asymp \frac{(\log q_{m+1})^B}{q_m},$$

*where  $K \asymp L$  denotes  $K = O(L)$  and  $L = O(K)$ .*

*Proof.* This is equation (6.3) of [2], p. 22.  $\square$

**Lemma 3.6.** *Let  $\alpha \in (0, 1) \setminus \mathbb{Q}$ . There are constants  $c_2, c_3 > 0$ , such that*

$$|g(\alpha)| \leq c_2 c(\alpha, +\infty) + c_3.$$

*Proof.* We closely follow [2], p. 65. By partial summation, we obtain:

$$g(\alpha) = \sum_{n \geq 1} \frac{\tau(n)}{n} \sin(2\pi n \alpha) = \int_1^{+\infty} Z_\tau(t, t; \alpha) \frac{dt}{t^2}$$

$$= \sum_{m \geq 1} \left( \int_{\xi_m}^{\xi_{m+1}} Z_\tau(t, t; \alpha) \frac{dt}{t^2} \right).$$

By equation 11.5 of [2], p. 65, we have

$$\int_{\xi_m}^{\xi_{m+1}} Z_\tau(t, t; \alpha) \frac{dt}{t^2} = \frac{1}{2} \pi \operatorname{sgn}(\varepsilon_m) \frac{\log q_{m+1}}{q_m} + O\left(\frac{1}{q_m^{1-1/B}} + \int_{\xi_m}^{\xi_{m+1}} \frac{dt}{t(\log t)^A}\right),$$

where  $A$  is fixed, but arbitrarily large.  
Therefore

$$\begin{aligned} g(\alpha) &= \int_1^{+\infty} Z_\tau(t, t; \alpha) \frac{dt}{t^2} \\ &\leq c_2 \sum_{m \geq 1} \frac{\log q_{m+1}}{q_m} + \sum_{m \geq 1} q_m^{1-1/B} + \int_1^{+\infty} \frac{dt}{t(\log t)^A} \\ &\leq c_2 c(\alpha, +\infty) + c_3, \end{aligned}$$

since the sequence  $(q_m)_{m \geq 1}$  is growing exponentially and the integral converges if  $A > 1$ . This completes the proof.  $\square$

*Proof of Theorem 1.1.* Let  $L \in \mathbb{N}$  and assume that  $\alpha$  satisfies  $(**)$  (Théorème 4.4. of [2]) and  $|g(\alpha)| \geq 4L$ .

We apply Lemmas 2.9 and 3.6 and obtain

$$\text{meas}\{\alpha : |g(\alpha)| \geq yL\} \leq \exp(-c_1 yL).$$

Therefore

$$\begin{aligned} \int_0^1 |g(\alpha)|^L d\alpha &\leq \sum_{j \geq 0} ((2^{j+1}L)^L \text{meas}\{\alpha : 2^j L \leq |g(\alpha)| \leq 2^{j+1}L\}) \\ &\leq \sum_{j \geq 0} (2^{j+1}L)^L \exp(-c_1 2^j L) \leq C_0^L L^L. \end{aligned}$$

$\square$

However,

$$\begin{aligned} H_k &= \int_0^1 \left( \frac{g(x)}{2\pi} \right)^{2k} dx = (2\pi)^{-2k} \int_0^1 g(x)^{2k} dx \\ &\leq (2\pi)^{-2k} C_0^{2k} (2k)^{2k} \\ &= \left( \frac{C_0}{2\pi} \right)^{2k} (2k)^{2k}, \end{aligned}$$

because of Theorem 1.1 with  $L = 2k$ ,  $k \in \mathbb{N}$ .

Also,

$$(2k)^{2k} \leq (2k)! 3^{2k},$$

for  $k \geq k_0$ , for some  $k_0 \in \mathbb{N}$ .

Hence

$$\begin{aligned} \frac{H_k}{(2k)!} &\leq \left( \frac{C_0}{2\pi} \right)^{2k} 3^{2k} \\ &= \left( \frac{3 C_0}{2\pi} \right)^{2k}, \end{aligned}$$

for  $k \geq k_0$ , for some  $k_0 \in \mathbb{N}$ .

Hence, the radius of convergence of the series

$$\sum_{k \geq 0} \frac{H_k}{(2k)!} x^k$$

is positive.

For the proof of Theorem 1.2, the following definitions and lemmas will be used.

**Definition 3.7.** For  $k \in \mathbb{N} \cup \{0\}$  we set

$$I := I(k) = [0, e^{-2k}] \quad \text{and} \quad l_0 := l_0(k) = e^{2k}.$$

We fix  $\delta > 0$  arbitrarily small and set

$$g_1(\alpha) := \sum_{l \leq l_0^{1-2\delta}} \frac{B(l\alpha)}{l}, \quad g_2(\alpha) := \sum_{l_0^{1-2\delta} < l \leq l_0^{1+2\delta}} \frac{B(l\alpha)}{l}, \quad g_3(\alpha) := \sum_{l > l_0^{1+2\delta}} \frac{B(l\alpha)}{l},$$

where  $B(u) = 1 - 2\{u\}$ ,  $u \in \mathbb{R}$ .

In the sequel, we assume  $k \geq k_0$ , where  $k_0 \in \mathbb{N}$ , sufficiently large.

**Lemma 3.8.** We have

$$g(\alpha) = g_1(\alpha) + g_2(\alpha) + g_3(\alpha),$$

for every  $\alpha \in \mathbb{R}$ .

*Proof.* It is obvious by the definition of  $g(\alpha)$ ,  $g_1(\alpha)$ ,  $g_2(\alpha)$ ,  $g_3(\alpha)$ . □

**Lemma 3.9.** For  $\alpha \in I$ , we have

$$g_1(\alpha) \geq (1 - 8\delta)2k,$$

for  $k \in \mathbb{N} \cup \{0\}$ .

*Proof.* For  $\alpha \in I$ ,  $l \leq l_0^{1-2\delta}$  we have  $l\alpha \leq \delta$  and therefore

$$B(l\alpha) \geq 1 - 4\delta$$

because of Definition 3.7. Thus

$$g_1(\alpha) \geq (1 - 4\delta) \sum_{l \leq l_0^{1-2\delta}} \frac{1}{l}.$$

From the formula

$$\sum_{m \leq u} \frac{1}{m} = \log u + O(1) \quad (u \rightarrow +\infty),$$

we have

$$g_1(\alpha) \geq (1 - 8\delta)2k.$$

□

**Lemma 3.10.** It holds

$$|g_2(\alpha)| \leq 16\delta k,$$

for  $k \in \mathbb{N} \cup \{0\}$  and sufficiently small  $\delta > 0$ .

*Proof.* We have

$$\begin{aligned} |g_2(\alpha)| &\leq \sum_{l_0^{1-2\delta} < l \leq l_0^{1+2\delta}} \frac{1}{l} \leq 2 (\log(l_0^{1+2\delta}) - \log(l_0^{1-2\delta})) \\ &\leq 16\delta k. \end{aligned}$$

□



**Lemma 3.11.** *For all  $\alpha \in I$  that do not belong to an exceptional set  $\mathcal{E}$  with measure*

$$\text{meas}(\mathcal{E}) \leq e^{-2k(1+\delta)},$$

*we have*

$$|g_3(\alpha)| \leq \delta k.$$

*Proof.* The function  $g_3$  has the Fourier expansion:

$$g_3(\alpha) = \sum_{l > l_0^{1+2\delta}} c(l) e(l\alpha),$$

where  $c(l) = O(l^{-1+\epsilon})$  for  $\epsilon$  arbitrarily small, by Lemma 5.6 of [5].  
By Parseval's identity we have

$$\int_0^1 g_3(\alpha)^2 d\alpha = \sum_{l > l_0^{1+2\delta}} c(l)^2 = O\left(\sum_{l > l_0^{1+2\delta}} l^{-2+2\epsilon}\right) = O\left(l_0^{-1-3\delta/2}\right).$$

Let

$$\mathcal{E} = \{\alpha : |g_3(\alpha)| > \delta k\}.$$

Then

$$\begin{aligned} (\text{meas}(\mathcal{E}))(\delta k)^2 &\leq \int_{\mathcal{E}} g_3(\alpha)^2 d\alpha \leq \int_0^1 g_3(\alpha)^2 d\alpha \\ &= O\left(l_0^{-1-3\delta/2}\right). \end{aligned}$$

Therefore

$$\text{meas}(\mathcal{E}) \leq O\left((\delta k)^{-2} l_0^{-1-3\delta/2}\right) = O\left(e^{-2k(1+\delta)}\right).$$

This completes the proof of the Lemma. □

*Proof of Theorem 1.2.*

By Lemmas 3.9, 3.10 and 3.11, we have

$$|g(\alpha)| \geq |g_1(\alpha)| - |g_2(\alpha)| - |g_3(\alpha)| \geq (1 - 20\delta)2k,$$

for all  $\alpha \in I$  except for those values of  $\alpha$  that belong to an exceptional set

$$\mathcal{E}(I) := \mathcal{E} \cap I \subset I$$

with

$$\text{meas}(\mathcal{E}(I)) \leq \frac{1}{2}|I|,$$

where  $|I|$  stands for the length of  $I$ . Hence, we obtain

$$\begin{aligned} H_k &= \int_0^1 \left(\frac{g(\alpha)}{\pi}\right)^{2k} d\alpha \geq \frac{1}{2}|I| \left(\frac{1-20\delta}{\pi} 2k\right)^{2k} \\ &= \frac{1}{2} e^{-2k} \exp(2k \log 2k) \left(\frac{1-20\delta}{\pi}\right)^{2k}. \end{aligned}$$

By Stirling's formula we have

$$(2k)! \geq \exp(2k \log 2k) \exp(-(1-\delta)2k)$$

and therefore

$$\frac{H_k}{(2k)!} \geq \frac{1}{2} e^{2\delta k} \left(\frac{1-20\delta}{\pi}\right)^{2k}.$$

Since  $\delta > 0$  can be fixed arbitrarily small, we have

$$\limsup_{k \rightarrow +\infty} \left( \frac{H_k}{(2k)!} \right)^{1/k} \geq \frac{1}{\pi^2}.$$

Therefore, the series

$$\sum_{k \geq 0} \frac{H_k}{(2k)!} x^k$$

diverges for  $|x| > \pi^2$ , where  $x \in \mathbb{C}$ . This completes the proof of Theorem 1.2.  $\square$

#### 4. THE DISTRIBUTION OF THE COTANGENT SUMS $c_0\left(\frac{r}{b}\right)$

We now give a simpler proof of Theorem 5.2 of [5] regarding the equidistribution of  $c_0(r/b)$  for fixed large positive integer values of  $b$  and  $A_0b \leq r \leq A_1b$ , where  $1/2 < A_0 < A_1 < 1$ . We need the following Lemmas and Definitions from [1].

**Lemma 4.1.** *Let  $\mu$  be a probability measure on the line having finite moments*

$$\alpha_k = \int_{-\infty}^{+\infty} x^k \mu(dx)$$

*of all orders. If the power series*

$$\sum_{k \geq 1} \alpha_k \frac{r^k}{k!}$$

*has a positive radius of convergence, then  $\mu$  is the only probability measure with the moments  $\alpha_1, \alpha_2, \dots$*

*Proof.* This is Theorem 30.1 of [1], pp. 388-389.  $\square$

**Definition 4.2.** *A probability measure satisfying the conclusion of Lemma 4.1 is said to be **determined by its moments**.*

**Definition 4.3.** *A sequence  $(F_n)_{n \geq 1}$  of distribution functions is said to **converge weakly** to the distribution function  $F$  (denoted by  $F_n \Rightarrow F$ ) if*

$$\lim_{n \rightarrow +\infty} F_n(x) = F(x)$$

*for every point  $x$  of continuity of  $F(x)$ .*

*A sequence  $(X_n)_{n \geq 1}$  of random variables is said to **converge in distribution** (or **in law**) towards a random variable  $X$  (denoted by  $X_n \Rightarrow X$ ) with distribution function  $F$ , if and only if  $F_n \Rightarrow F$ , that is  $X_n \Rightarrow X \Leftrightarrow F_n \Rightarrow F$ .*

**Lemma 4.4.** *For a sequence  $(X_n)_{n \geq 1}$  of random variables and a random variable  $X$ , we have  $X_n \Rightarrow X$  if and only if*

$$\lim_{n \rightarrow +\infty} P[X_n \leq x] = P[X \leq x]$$

*for every  $x \in \mathbb{R}$ , such that  $P[X = x] = 0$ .*

*Proof.* This follows immediately from Definition 4.3.  $\square$

**Lemma 4.5.** Suppose that the distribution of  $X$  is determined by its moments and that the  $X_n$  have moments of all orders, as well as

$$\lim_{n \rightarrow +\infty} E(X_n^r) = E(X^r)$$

for  $r = 1, 2, 3, \dots$ . Then  $X_n \Rightarrow X$ .

*Proof.* This is Theorem 30.2 of [1], p. 390. □

We now recall the Definition 5.1 and Theorem 5.2 from [5].

**Definition 4.6.** For  $z \in \mathbb{R}$ , let

$$F(z) = \text{meas}\{\alpha \in [0, 1] : g(\alpha) \leq z\},$$

where “meas” denotes the Lebesgue measure,

$$g(\alpha) = \sum_{l=1}^{+\infty} \frac{1 - 2\{l\alpha\}}{l}$$

and

$$C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) : \forall \epsilon > 0, \exists a \text{ compact set } \mathcal{K} \subset \mathbb{R}, \text{ such that } |f(x)| < \epsilon, \forall x \notin \mathcal{K}\}.$$

**Theorem 4.7.** i)  $F$  is a continuous function of  $z$ .

ii) Let  $A_0, A_1$  be fixed constants, such that  $1/2 < A_0 < A_1 < 1$ . Let also

$$H_k = \int_0^1 \left( \frac{g(x)}{\pi} \right)^{2k} dx,$$

where  $H_k$  is a positive constant depending only on  $k$ ,  $k \in \mathbb{N}$ .

There is a unique positive measure  $\mu$  on  $\mathbb{R}$  with the following properties:

(a) For  $\alpha < \beta \in \mathbb{R}$  we have

$$\mu([\alpha, \beta]) = (A_1 - A_0)(F(\beta) - F(\alpha)).$$

(b)

$$\int x^k d\mu = \begin{cases} (A_1 - A_0)H_{k/2}, & \text{for even } k \\ 0, & \text{otherwise.} \end{cases}$$

(c) For all  $f \in C_0(\mathbb{R})$ , we have

$$\lim_{b \rightarrow +\infty} \frac{1}{\phi(b)} \sum_{\substack{r : (r,b)=1 \\ A_0 b \leq r \leq A_1 b}} f\left(\frac{1}{b} c_0\left(\frac{r}{b}\right)\right) = \int f d\mu,$$

where  $\phi(\cdot)$  denotes the Euler phi-function.

We now state and give a new proof of a special case of Theorem 4.7 (c) from which the complete Theorem 4.7 follows by the definition of the abstract Lebesgue integral.

**Theorem 4.8.** Let  $A_0, A_1$  be fixed constants, such that  $1/2 < A_0 < A_1 < 1$ , then we have for  $\alpha < \beta \in \mathbb{R}$  :

$$\begin{aligned} & \lim_{b \rightarrow +\infty} \frac{1}{\phi(b)} \left| \left\{ r : (r, b) = 1, A_0 b \leq r \leq A_1 b, \alpha b \leq c_0\left(\frac{r}{b}\right) \leq \beta b \right\} \right| \\ &= (A_1 - A_0)(F(\beta) - F(\alpha)). \end{aligned}$$

*Proof.* Let  $(b_n)_{n \geq 1}$  be a sequence of positive integers with  $b_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ . We set

$$X_n = \frac{1}{b_n} c_0 \left( \frac{r}{b_n} \right)$$

and consider  $X_n$  as a random variable on the probability space

$$\Omega_n = \{r : (r, b_n) = 1, A_0 b_n \leq r \leq A_1 b_n\}$$

with the counting measure

$$\mu_n(\mathcal{E}) = \frac{|\mathcal{E}|}{|\Omega_n|}$$

for all  $\mathcal{E} \subset \Omega_n$ .

By Lemma 5.13 of [5], we have

$$\lim_{n \rightarrow +\infty} \mu_n([\alpha, \beta]) = (A_1 - A_0)(F(\beta) - F(\alpha))$$

for all  $\alpha < \beta \in \mathbb{R}$ .

By Theorem 1.1, Lemma 4.1 and Definition 4.2 the measure  $\mu$  given by

$$\mu([\alpha, \beta]) = (A_1 - A_0)(F(\beta) - F(\alpha)),$$

is determined by its moments. By Theorem 4.7 we have

$$\lim_{n \rightarrow +\infty} E(X_n^r) = E(X^r).$$

Thus, Lemma 4.5 implies  $X_n \Rightarrow X$ , where  $X = g(\alpha)$  is a random variable on the probability space  $[0, 1]$ . Since  $F$  is a continuous function by Theorem 5.2(i) of [5], the claim of Theorem 4.8 follows.  $\square$

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